Problem Sheet 2

Exercise 2.1.

Let $\mathcal{X}$ be a complete separable metric space. Prove that any probability measure on $\mathcal{X}$ is tight. *Hint: A complete, totally bounded set is compact.*

Exercise 2.2.

Let μ, ν be two probability measures supported on an interval $[a, b] \subset \mathbb{R}$. Show that moments separate the measures, that is if for every non-negative integer k we have that

$$\int_a^b x^k d\mu = \int_a^b x^k d\nu$$

then $\mu = \nu$.

Exercise 2.3.

Let μ, ν be two probability measures on $\mathbb{R}$ such that for every non-negative integer k, we have that

$$\int_{\mathbb{R}} x^k d\mu = \int_{\mathbb{R}} x^k d\nu =: \alpha_k.$$

In addition suppose that there exists an $r > 0$ such that for all $s \in [0, r]$, the infinite sum

$$\sum_{k=1}^{\infty} \frac{\alpha_k r^k}{k!}$$

is well defined in $\mathbb{R}$. Demonstrate the following:

1. Defining $\beta_k := \int_{\mathbb{R}} |x|^k d\mu$, we have that for any $s \in (0, r)$, $\frac{\beta_k s^k}{k!} \to 0$ as $k \to \infty$;
2. You are given that for any $x \in \mathbb{R}$,

   $$\left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \leq \frac{|x|^{n+1}}{(n+1)!}. \tag{1}$$

   Denote the characteristic function of μ by ϕ, that is for any $t \in \mathbb{R}$,

   $$\phi(t) := \int_{\mathbb{R}} e^{itx} d\mu.$$

   Then for any $t \in \mathbb{R}$ and $h \in \mathbb{R}$ with $|h| < r$,

   $$\phi(t+h) = \sum_{k=0}^{\infty} \frac{h^k}{k!} \int_{\mathbb{R}} (ix)^k e^{itx} d\mu.$$

3. Further denote $\left(\frac{d}{dt}\right)^k \phi = \phi^{(k)}$. Then

   $$\phi^{(k)}(t) = \int_{\mathbb{R}} (ix)^k e^{itx} d\mu.$$

   Hint: Use (1) with $n = 1$.

4. Let ψ denote the characteristic function of ν. Show that $\phi = \psi$ and hence $\mu = \nu$.

Exercise 2.4.

Answer the following:

1. Is the space of continuous functions $f : \mathcal{X} \to \mathbb{R}$ measure separating?
2. Give sufficient conditions for the space of continuous and compactly supported functions $f : \mathcal{X} \to \mathbb{R}$ to be measure separating.

Exercise 2.5

Let $\mathcal{X}$ be a normed vector space. Consider the dual space $\mathcal{X}^*$ of all continuous linear functionals $f : \mathcal{X} \to \mathbb{R}$. Prove that $\mathcal{X}^*$ separates points.

Exercise 2.6

Let M be a collection of functions $f : \mathcal{X} \to \mathbb{R}$. Define M^2 to be the collection of functions $\phi : \mathcal{X}^2 \to \mathbb{R}$ given by

$$M^2 := \{\phi : \phi(x, y) = f(x)g(y) \,\text{some}\, f, g \in M\}\,.$$

Suppose that M separates points in $\mathcal{X}$.

1. Does M^2 separate points in $\mathcal{X}^2$?
2. Suppose that M is an algebra of functions. Construct a larger algebra H, that is $f \in M$ implies that $f \in H$, such that H^2 separates points in $\mathcal{X}^2$.